

# Hadron resonances, large $N_c$ , and the half-width rule\* \*\*

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We suggest using the half-width rule to make an estimate of the  $1/N_c$  errors in hadronic models containing resonances. We show simple consequences ranging from the analysis of meson Regge trajectories, the hadron resonance gas at finite temperature and generalized hadronic form factors.

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## 1. Introduction

The Particle Data Group (PDG) tables [1] represent the nowadays consensus of the particle spectrum, and it is quite legitimate to ask the question

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on the completeness or redundancy of the states listed there. The quantum numbers accommodated by the quark model for the u,d,s flavored mesons,  $n^{2S+1}L_J$  furnishes, till now, a complete commuting set of observables. On the other hand, since most of these states are unstable, a thorough understanding of the physics summarized by the PDG is related to the concept of a resonance.

However, the hadronic resonances are never observed directly but through their decay channels, and the corresponding cross section also depends on the particular production process. The standard and unique quantum mechanical definition of a resonance is via a pole in the second Riemann sheet of the complex- $s$  plane in a scattering amplitude containing such a resonance, although the (complex) residue depends on the process. This definition has the advantage of being quite universal regarding the pole position, but can only be applied if the amplitude can be analytically continued in a reliable way. Indeed, complex energies cannot be measured experimentally nor simulated by lattice QCD calculations, and basically an extrapolation is needed; a potentially uncontrolled arbitrary procedure [2]. Thus, in order to deduce these poles reliably, one must either have narrow resonances, small backgrounds, or accurate amplitudes, requirements which are rarely met in the PDG compilation [1].

There are other and more handy definitions which apply to the physical and real energy, such as the maximum in the speed plot, the time delay, or the popular Breit-Wigner definition. While all these definitions should naturally merge in the limit of narrow resonances, the finite widths build systematic differences which introduce some inherent dependence on the background. The upshot of the present discussion is that one should be concerned with i) what is the right value to quote and ii) to what confidence level can different values be considered as compatible. Again, the PDG compilation incorporates different processes which quite often rely on models or parameterizations.

## 2. Large $N_c$ and the half-width rule

A useful observation is that in the large  $N_c$  limit [3, 4] one has  $\Gamma/M = \mathcal{O}(N_c^{-1})$  and one finds [5]  $\Gamma/M = 0.12(8)$  both for mesons and baryons composed of the light u,d,s quarks and listed by the PDG [1]. Most mesonic and baryonic resonances stem from the  $\bar{q}q$  and  $qqq$  bound states which become unstable once they are allowed to decay in the continuum. We suggest that the maximum level of discrepancy in quoting resonance mass parameters should just be compatible with its own width, namely the interval  $M_R \pm \Gamma_R/2$ .

A model-independent way of looking at resonances in QCD is by consid-

ering the two-point correlation functions. Actually, in the quenched approximation one may treat them as standard bound states. Consider for instance the case of the  $\rho$ -meson, which is obtained as a  $\bar{q}q$  state from the vector-vector correlation function. The Lehman representation of the resonance two-point function is

$$D(s) = \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{\mu^2 - s - i0^+}, \quad (1)$$

suggesting a probabilistic interpretation of the line shape

$$P(\mu) = Z\rho(\mu) \quad (2)$$

as a function of the mass  $\mu$ . For a Breit-Wigner shape we have

$$D_{\text{BW}}(\mu) = \frac{1}{\mu^2 - M^2 - i\Gamma\mu} \rightarrow P_{\text{BW}}(\mu) = \frac{1}{\pi} \frac{2\Gamma\mu^2}{(\mu^2 - M^2)^2 + \Gamma^2\mu^2}. \quad (3)$$

The random implementation for a given distribution is obtained by inverting the relation

$$P(\mu)d\mu = dz \quad (4)$$

with  $z \in U[0, 1]$  denoting a uniformly distributed variable<sup>1</sup>. The half-width rule (HWR) consists of treating the resonance mass as a random variable and propagating its effect to all observables

### 3. Mesonic Regge Trajectories

Long ago it was suggested [6] that linear confinement in quark models implies the mesons *radial* Regge trajectories of the form generalizing the Chew-Frautchi plots for the angular momentum [7]. The analysis of Ref. [8] of  $M_n^2 = M_0^2 + \mu^2 n$  gave  $\mu^2 = 1.25(15) \text{ GeV}^2$ . The HWR amounts to minimize [9, 10]

$$\chi^2 = \sum_n \left( \frac{M_n^2 - M_{n,\text{exp}}^2}{\Gamma_n M_n} \right)^2, \quad (5)$$

The construction of these trajectories requires a choice on the possible quantum-number assignments. Following [10], the global result for the non-strange mesons was found

$$M^2 = 1.38(4)n + 1.12(4)J - 1.25(4). \quad (6)$$

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<sup>1</sup> One can also use Gaussian variables for  $\mu$  which have shorter tails.

Several models [11,12], including holographic approaches [13], assume a universal radial and angular momentum slope, i.e., an exact  $(n+J)$ -dependence. As we can see, there seems to be a significant deviation from this universality, similarly as in the relativistic quark model [14].

A global fit which does not require a selection of states, but assumes  $\bar{q}q$  completeness, considers the staircase function for the mesons, defined as

$$N_{\text{mesons}}(M) = \sum_{nLSJf,f'} (2S+1)\Theta(M^2 - an - bJ - c_{f,\bar{f}}) \quad (7)$$

where the spin-orbit and tensor force effects are neglected. Within the mass range  $0.5\text{GeV} \leq M \leq 1.85\text{GeV}$  and using  $\Delta M = 10\text{MeV}$  bins, it yields a flavor-dependent log-fit with  $a = 1.40\text{GeV}^2$ ,  $b = 1.10\text{GeV}^2$ ,  $c_{n\bar{n}} = -1.23\text{GeV}^2$ ,  $c_{n\bar{s}} = c_{s\bar{n}} = -0.40\text{GeV}^2$  and  $c_{s\bar{s}} = -0.78\text{GeV}^2$ , in good agreement with the non-strange single state determination and despite the fact that the degeneracy plays a role in the fit.

#### 4. Hadronic density of states

A further direct application of the HWR concerns the analysis of the cumulative hadron number

$$N(M) = \sum_i g_i \Theta(M - M_i) \quad (8)$$

where  $M_i$  are the individual hadronic masses and  $g_i$  are the corresponding spin degeneracies (particles and antiparticles are counted separately). This yields a staircase function which is presented in Fig. 1 when all the PDG hadrons [1] with the light  $u, d, s$  quarks are included. The exponential growth of  $N(M) \sim Ae^{M/T_H}$  can be distinctly seen, although, as pointed out in Ref. ([15–17]), a pre-exponent power cannot be extracted from the data. As seen in Fig. 1 the effect of taking into account the resonance uncertainty naturally smooths the data and provides an estimate of the finite width corrections, hence allowing an error analysis. Using the BW distribution for the widths we get  $T_H = 300(75)\text{MeV}$  and  $A = 1.758(2)$  with a  $\chi^2/\text{d.o.f.} = 0.92$  in the range  $0.5\text{GeV} \leq M \leq 1.85\text{GeV}$ .

Likewise, the trace anomaly in QCD has been calculated on the lattice [18] and below the cross-over transition to the quark-gluon plasma it can be represented in the Hadron Resonance Gas as follows

$$\Delta = \frac{\epsilon - 3p}{T^4} = \frac{1}{T^4} \int_0^\infty dM \frac{dN(M)}{dM} \int \frac{d^3k}{(2\pi)^3} \frac{(E_k - \vec{k} \cdot \nabla_k E_k)}{e^{E_k/T} \pm 1} \quad (9)$$

where  $E_k = \sqrt{k^2 + M^2}$  and  $\pm$  corresponds to Fermions/Bosons. As seen in Fig. 1, the half-width rule provides an error estimate for  $\Delta$  in the HRG which compares favorably with the lattice data of the Wuppertal group [18].

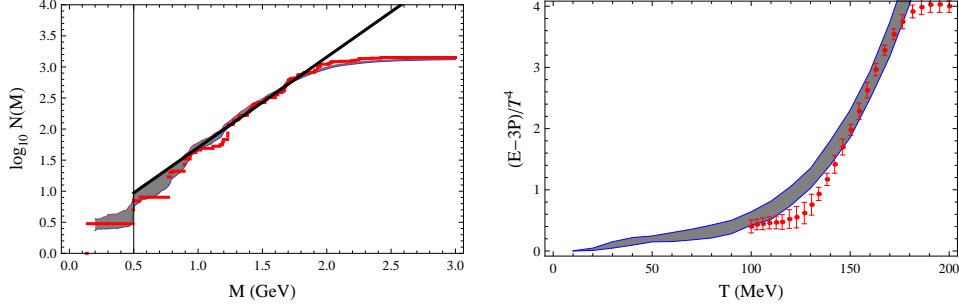


Fig. 1. Left: Cumulative number of hadrons from the PDG [1] when resonances are represented as random BW variables  $M_R \pm \Gamma_R/2$  as a function of the energy. We also plot the exponential spectrum  $N(M) = Ae^{M/T_H}$  with the Hagedorn temperature given by  $T_H = 300(75)\text{MeV}$ . Right: Trace anomaly of the Hadron Resonance Gas implementing the half-width rule compared to the lattice data of Ref. [18]

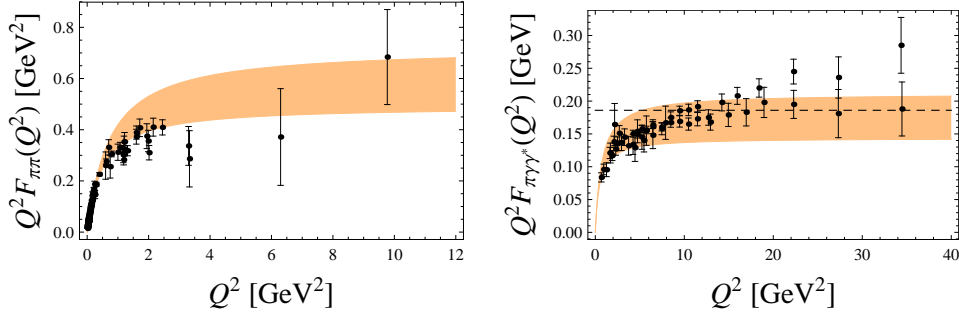


Fig. 2. The monopole form factors  $F(Q^2) = m_\rho^2/(m_\rho^2 + Q^2)$  gaussian sampled with the HWR,  $\pm\Gamma_\rho/2 = \pm 0.75\text{MeV}$ . Left: Charge pion form factor,  $F_{\gamma^*\pi\pi}(Q^2) = F(Q^2)$ . Right: Transition form factor,  $F_{\pi\gamma\gamma^*}(Q^2) = F(Q^2)/(4\pi f_\pi)$

## 5. Hadronic form factors

As a final example of the HWR let us consider hadronic generalized form factors as analyzed recently [19]. The well-known fact is that in the large- $N_c$  limit of QCD the generalized hadronic form factors, probing bilinear  $\bar{q}q$  operators with given  $J^{PC}$  quantum numbers, feature generalized meson dominance of  $\bar{q}q$  states with the same quantum numbers,

$$\langle A(p') | J(0) | \langle B(p) \rangle \sim \sum_n \frac{c_n^{AB} m_n^2}{m_n^2 - t}, \quad (10)$$

where  $m_n$  are the meson masses and  $c_n^{AB}$  the suitable couplings. Thus generalized form factors at some finite momentum transfer essentially measure

the masses of the lowest lying mesons in the given channel. Saturating with the minimum number of mesons to yield the known high  $Q^2$  and applying the HWR produces an error band estimate which we show in Fig. (2) for the pion electromagnetic and transition form factors. Refinements and further pion and nucleon form factors are presented in [19]. While large- $N_c$  behavior of hadronic quantities provides a unique fingerprint of QCD. Quark-Hadron Duality allows to sidestep the difficult problems by imposing the short-distance constraints. Using a minimum number of resonances leads to a sensible parameter reduction. Errors based on the half-width rule provide a reasonable and large  $N_c$  motivated estimate on the uncertainties of form factors in the space-like region.

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